Synchronization of fractional order chaotic systems

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The chaotic dynamics of fractional order systems began to attract much attention in recent years. In this Brief Report, we study the master-slave synchronization of fractional order chaotic systems. It is shown that fractional order chaotic systems can also be synchronized.

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Fractional calculus is a 300-year-old topic. Although it has a long mathematical history, the applications of fractional calculus to physics and engineering are just a recent focus of interest [1,2]. Many systems are known to display fractional order dynamics, such as viscoelastic systems [3–5], dielectric polarization [6], electrode-electrolyte polarization [7], and electromagnetic waves [8]. More recently, many authors began to investigate the chaotic dynamics of fractional order dynamical systems [9–17]. In Ref. [9], it has been shown that the fractional order Chua's system of order as low as 2.7 can produce a chaotic attractor. In Ref. [10], it has been shown that nonautonomous Duffing systems of order less than 2 can still behave in a chaotic manner. In Ref. [11], chaotic behaviors of the fractional order "jerk" model was studied, in which chaotic attractor was obtained with system orders as low as 2.1, and in Ref. [12] the chaos control of this fractional order chaotic system was reported. In Ref. [13], chaotic behavior of the fractional order Lorenz system was studied, but unfortunately, the results presented in that paper are not correct. In Refs. [14] and [15], bifurcation and chaotic dynamics of the fractional order cellular neural networks were studied. In Ref. [16], chaos and hyperchaos in the fractional order Rössler equations were studied, in which we showed that chaos can exist in the fractional order Rössler equation with order as low as 2.4, and hyperchaos exists in the fractional order Rössler hyperchaos equation with order as low as 3.8. In Ref. [17], we have studied the chaotic behavior and its control in the fractional order Chen system. In Ref. [18], the author presents a broad review of existing models of fractional kinetics and their connection to dynamical models, phase space topology, and other characteristics of chaos.

On the other hand, synchronization of chaotic systems has attracted much attention [19] since the seminal paper by Pecora and Carroll [20]. In this Brief Report, we study the synchronization of fractional order chaotic systems. The analysis of fractional order systems is by no means trivial. So, we will numerically investigate this topic here.

There are many definitions of fractional derivatives [1]. Perhaps the best known one is the Riemann-Liouville definition, which is given by

$$\frac{d^\alpha f(t)}{dt^\alpha}=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau \qquad (1)$$

where $\Gamma(\cdot)$ is the gamma function and $n-1\leq\alpha<n$. The geometric and physical interpretation of the fractional derivatives was given in Ref. [21]. Upon considering the initial conditions to be zero, the Laplace transform of the Riemann-Liouville fractional derivative is $L\{d^\alpha f(t)/dt^\alpha\}=s^\alpha L\{f(t)\}$. So, the fractional integral operator of order "α" can be represented by the transfer function $F(s)=1/s^\alpha$.

The standard definition of the fractional differintegral do not allow direct implementation of the fractional operators in time-domain simulations. An efficient method to circumvent this problem is to approximate the fractional operators by using the standard integer order operators. In the following simulations, we will use the approximation method proposed in Ref. [22], which was also adopted in [9,11,14–17]. In Table 1 of Ref. [9], the authors gave approximations for $1/s^q$ with $q=0.1$–0.9 in step 0.1 with errors of approximately 2 dB. We will use these approximations in our following simulations.

Consider the master-slave synchronization scheme of two autonomous n-dimensional fractional order chaotic systems

$$M\colon \frac{d^\alpha x}{dt^\alpha}=f(x),$$

$$S\colon \frac{d^\alpha y}{dt^\alpha}=f(y)+c\Gamma(x-y) \qquad (2)$$

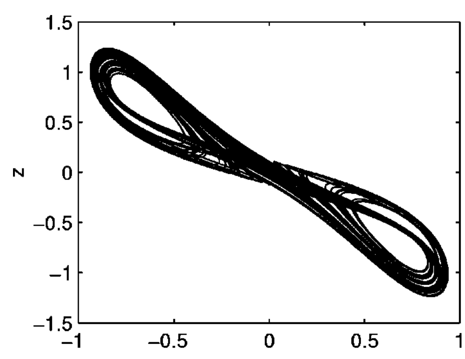

FIG. 1. Phase plot of the fractional order Chua's system with $\alpha=0.9$.

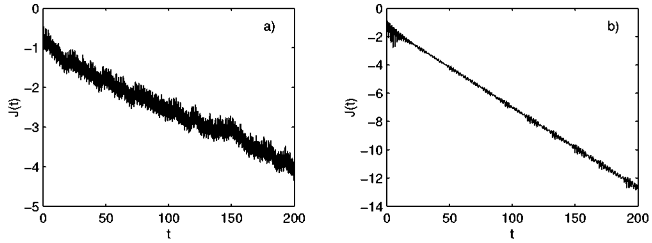

FIG. 2. Synchronization error of the fractional order Chua's systems with $\alpha=0.9$: (a) $c=4$, (b) $c=7$.

with the master system M and the slave system S. Where $\alpha>0$ is the fractional order, with which the individual dynamical systems are chaotic, $c>0$ is the coupling strength, and $\Gamma\in R^{n\times n}$ is a constant 0–1 matrix linking the coupling variables. For simplicity, we assume $\Gamma=\mathrm{diag}(r_1,r_2,\ldots,r_n)$ is a diagonal matrix. If there is a coupling between the ith state variable of the two coupled chaotic systems, then $r_i=1$; otherwise, $r_i=0$. Define the error signal as $e=x-y$, the aim of the synchronization scheme is to design the coupling strength such that $\|e(t)\|\to 0$ as $t\to\infty$. This scheme is similar to the master-slave synchronization of classical integer-order chaotic systems.

Next, we numerically study the synchronization of fractional order chaotic systems via two examples. We first consider the fractional order Chua's system [9]

$$\frac{d^\alpha x}{dt^\alpha}=a\left[y+\frac{x-2x^3}{7}\right],$$

$$\frac{d^\alpha y}{dt^\alpha}=x-y+z, \tag{3}$$

$$\frac{d^\alpha z}{dt^\alpha}=-\frac{100}{7}y,$$

when $\alpha\geqslant 0.9$, this system can produce chaotic solutions [9]. Particularly, when $\alpha=0.9$ and $a=12.75$, the fractional order Chua's system is chaotic. The phase plot of x and z is shown in Fig. 1.

We let $\Gamma=\mathrm{diag}(1,0,0)$, which implies that only the first variable x is used for coupling the two fractional order chaotic systems. To obtain a critical value of c to make the two systems synchronized, we continuously increase the coupling strength c, from $c=0$, in step 0.5. When $c<4$, no synchronous phenomenon is observed. When $c=4$, the curve of the synchronization error $J(t)=\log(\|e(t)\|)$ is shown in Fig. 2 (a), which indicates that the master-slave synchronization is achieved. In Fig. 2(b), we show the curve of the synchronization error when $c=7$, in which the synchronization effect is better than that of $c=4$.

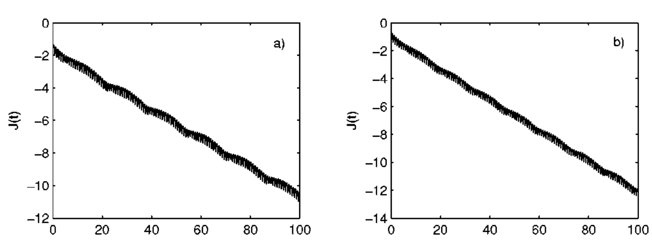

FIG. 3. Synchronization error of the integer order Chua's systems: (a) $c=4$, (b) $c=7$.

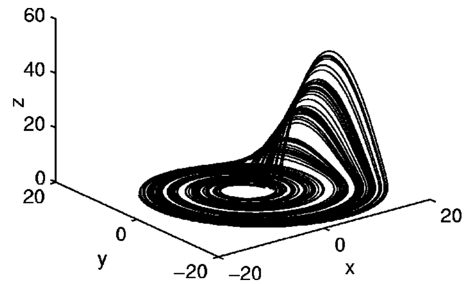

FIG. 4. Phase plot of the fractional order Rössler system with $\alpha=0.9$.

For the purpose of comparison, we also plot the curves of synchronization error of the integer order Chua's systems ($a=9.5$) in Fig. 3. Comparing Fig. 2 with Fig. 3, we can know that the synchronization rate of the fractional order Chua's systems is slower than its integer order counterpart.

We next consider the fractional order Rössler system [16]

$$\frac{d^\alpha x}{dt^\alpha}=-(y+z),$$

$$\frac{d^\alpha y}{dt^\alpha}=x+ay, \tag{4}$$

$$\frac{d^\alpha z}{dt^\alpha}=0.2+z(x-10),$$

when $\alpha=0.9$ and $a=0.4$, the above system is chaotic. The phase diagram of the chaotic attractor is shown in Fig. 4.

We also let $\Gamma=\mathrm{diag}(1,0,0)$, and do the similar simulations as in the above example. When the coupling strength $c=0.5$, the two fractional Rössler systems achieve synchronization. The curve of the synchronization error of the fractional order Rössler system is shown in Fig. 5(a). In Fig. 5(b), we also plot the curve of the synchronization error of the integer order Rössler systems ($a=0.165$). From Fig. 5,

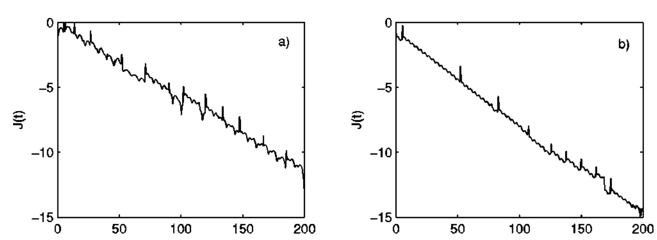

FIG. 5. The curves of synchronization error: (a) the fractional order Rössler systems with $\alpha=0.9$ and $c=0.5$; (b) the integer order Rössler systems with $c=0.5$.

we know that the synchronization rate of the fractional order Rössler systems is also slightly slower than its integer counterpart.

We have also tested the synchronization scheme (2) on several other fractional order chaotic systems [23]. Limited to the length of this Brief Report, we omit these results here.

In summary, in this Brief Report, we have studied the master-slave synchronization of fractional order chaotic systems. To the best of our knowledge, this is the first report on the synchronization of fractional order dynamical systems. We have shown that fractional order chaotic systems can be synchronized by utilizing the similar scheme as that of their integer order counterparts.

Future works regarding this topic include the investigation of some other types of synchronization of fractional order chaotic systems, such as the phase synchronization [24] and the projective synchronization [25], as well as the synchronization of fractional order hyperchaotic systems [16].

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